Translates of Multivariate Splines

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Dedicated to Alexander M. Ostrowski
on the occasion of his ninetieth birthday.

Submitted by Chandler Davis

ABSTRACT

We study spaces generated by translations of a fixed function over lattice points. We provide some new algebraic and approximation properties for these spaces which show their applicability for finite element analysis.

1. INTRODUCTION

Let $C_0(R^s)$ denote the class of continuous functions of compact support in R^s. With every $\phi \in C_0(R^s)$ we associate the linear space of all its translates over the lattice points in Z^s,

$$\mathbb{S}_\phi = \left\{ \sum_\alpha a_\alpha \phi(x-\alpha) : \alpha \in Z^s, a_\alpha \in R \right\}. \tag{1.1}$$

In this paper, we will study certain algebraic and approximation properties of the space $\mathbb{S}_\phi$.

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Our interest in these questions was stimulated by the recent interesting paper of de Boor and Höllig [3]. They studied this space only when ϕ is the so-called *box spline*. The box spline is a variation on the idea used to define the *multivariate B-spline* [7]. Both functions are special cases of the following geometric construction.

For any convex body $V \subseteq R^n$ and any set $K = \{x^1, \ldots, x^n\} \subset R^s$ we define the distribution

$$T_V(f|x^1,\ldots,x^n) = \int_V f\left(\sum_{i=1}^n \lambda_i x^i\right) d\lambda_1 \cdots d\lambda_n, \qquad f \in C_0^\infty(R^s). \quad (1.2)$$

When the vectors in K span R^s, $\langle K \rangle = R^s$, this distribution corresponds to a density function

$$T_V(f|x^1,\ldots,x^n) = \int_{R^s} M_V(x|x^1,\ldots,x^n) f(x)\, dx. \quad (1.3)$$

The special case $V = \{(\lambda_1,\ldots,\lambda_n)\colon \sum_{i=1}^n \lambda_i \leqslant 1,\ \lambda_i \geqslant 0\}$ (the standard n-simplex) gives the B-spline $M(x|0, x^1,\ldots,x^n)$ studied in [7]. Choosing V to be the n-cube, $V = [0,1]^n$, gives the box spline [3]. When $\langle K \rangle = R^s$ the box spline is a density function which we denote by $B(x|x^1,\ldots,x^n)$. The defining equation for this function is therefore

$$\int_{[0,1]^n} f\left(\sum_{i=1}^n \lambda_i x^i\right) d\lambda_1 \cdots d\lambda_n = \int_{R^s} f(x) B(x|x^1,\ldots,x^n)\, dx. \quad (1.4)$$

Many intriguing facts have been discovered about $B(x|x^1,\ldots,x^n)$ which indicate its importance in the theory of multivariate splines on a regular grid.

We will show here, under certain general circumstances to be described later, that the *translates of* $B(x|x^1,\ldots,x^n)$ *are linearly independent*. This was only known (in a stronger sense) in R^2 for a very special case studied in [4] (the "three direction mesh"). Primarily though we focus on the *simple* form of the Fourier transform of $B(x|x^1,\ldots,x^n)$. Specifically, from (1.3) we easily obtain

$$\hat{B}(x) = \int_{R^s} e^{-ix\cdot y} B(y)\, dy = \prod_{k=1}^n \frac{1 - e^{-ix^k\cdot x}}{ix^k\cdot x}. \quad (1.5)$$

Note that in the terminology of [6], $\hat{B}(x)$ is an entire function of affine lineage. We will show that many of the results in [3], established only for B, also hold for a wide class of functions. Their characteristic feature is the Fourier transform representation

$$\hat{\phi}(x) = \hat{\rho}_1(x \cdot x^1) \cdots \hat{\rho}_n(x \cdot x^n) \tag{1.6}$$

Equivalently, we may define ϕ as the distribution

$$\int_{R^n} f\left(\sum_{i=1}^{n} \lambda_i x^i \right) \rho_1(\lambda_1) \cdots \rho_n(\lambda_n)\, d\lambda_1 \cdots d\lambda_n.$$

This type of distribution has already appeared in [5].

The methods we use to study the space $\mathcal{S}_\phi$ for functions whose Fourier transform has the form (1.6) are quite different from those employed in [3]. Instead, we follow general methods already used by I. J. Schoenberg in his seminal paper [9], which were later significantly extended and refined by Strang and Fix [8]. They showed the importance of these ideas in the *finite element method*.

Actually, we will not specifically use any results of these authors. Our common ground is the use of the *Poisson summation formula*. This important identity allows us to improve upon the main results in [3].

2. POISSON'S SUMMATION FORMULA

The Poisson summation formula states that

$$\sum_\alpha \phi(\alpha) = \sum_\alpha \hat{\phi}(2\pi\alpha). \tag{2.1}$$

This formula is valid under various hypotheses on ϕ and $\hat{\phi}$. For us it is sufficient to note that (2.1) holds whenever ϕ and $\hat{\phi}$ are *rapidly decreasing* (see e.g. [12, p. 149]). In fact, we only require the following simple consequence of (2.1).

LEMMA 2.1. *Let $\phi \in C_0(R^s)$, and suppose $\hat{\phi}(2\pi\alpha) = 0$, $\alpha \in Z^s - \{0\}$. Then*

$$\hat{\phi}(0) = \sum_\alpha \phi(\alpha). \tag{2.2}$$

Proof. When $\hat{\phi}$ is rapidly decreasing we can apply (2.1) directly to obtain (2.2). Otherwise, we can mollify ϕ with, say, the Gaussian kernel,

$$\phi_\varepsilon(x) = G_\varepsilon * \phi, \qquad \varepsilon > 0.$$

Since $\hat{\phi}_\varepsilon(2\pi\alpha) = 0$, $\alpha \in Z^s - \{0\}$, we can apply (2.1) to ϕ_ε and then pass $\varepsilon \to 0^+$. ∎

REMARK 2.1. It should perhaps be pointed out that the hypotheses of Lemma 2.1 are in competition with each other. For instance, when $s = 1$ and $\operatorname{supp} \phi = \{x : \phi(x) \neq 0\} \subseteq [-\frac{1}{2}, \frac{1}{2}]$, then the only function which satisfies the hypotheses of this lemma is

$$\phi(t) = \phi(0)\chi_{[-\frac{1}{2},\frac{1}{2}]}(t).$$

The reason for this is that

$$\hat{\phi}(2\pi t) - \hat{\phi}(0)\frac{\sin \pi t}{\pi t}$$

is necessarily an entire function of exponential type $\leqslant \pi$ which vanishes at all the integers and which tends to zero as $t \to \infty$. Hence by a well-known result on entire functions (see e.g. [1, p. 156]) it must be identically zero.

Of course, if we allow the support of ϕ to be larger, then there are nontrivial ϕ's satisfying the conditions of Lemma 2.1. Specifically, we can choose $\phi = \psi * \chi_{[-\frac{1}{2},\frac{1}{2}]}$ where $\psi \in C_0(R^s)$.

We will use Π to denote all polynomials on $\mathbf{C}^s$, while Π_n will stand for all polynomials of total degree $\leqslant n$. It is also convenient for us to introduce the class

$$\mathcal{P}_\phi = \{p : p \in \Pi, (p(D)\hat{\phi})(2\pi\alpha) = 0, \alpha \in Z^s - \{0\}\}, \tag{2.3}$$

where, as usual, $p(D)$ is the constant coefficient differential operator induced by p. This set is somewhat difficult for us to work with. We will instead deal with subspaces of $\mathcal{P}_\phi$ which are *affinely invariant*. A set $\mathcal{P}$ is said to be affinely invariant if

(1) $\mathcal{P}$ is a subspace of Π;
(2) whenever $p \in \mathcal{P}$, $p(ax + y) \in \mathcal{P}$ for $a \in \mathbf{C}$, $y \in R^s$.

For later reference, we record some elementary facts about affinely invariant subspaces.

LEMMA 2.2. *Let $\mathcal{P}$ be a nontrivial affinely invariant subspace. Then:*

(i) *For every $y \in R^s$, $p \in \mathcal{P}$ the directional derivative $D_y p \in \mathcal{P}$.*

(ii) $1 \in \mathcal{P}$.

(iii) *Whenever $p \in \mathcal{P}$ has a representation $p = q_1 + \cdots + q_m$ where each q_i is a homogeneous polynomial with distinct orders, then $q_i \in \mathcal{P}$, $i = 1, \dots, m$.*

REMARK 2.2. This last property means that $\mathcal{P}$ can be "graded" by its homogeneous members.

Also, note that when $s = 1$, $\mathcal{P}$ must be all polynomials of degree $\leqslant n$ for some n (either finite or infinite).

The next result shows the usefulness of this notion.

PROPOSITION 2.1. *Suppose $\mathcal{P}$ is an affinely invariant subspace of $\mathcal{P}_\phi$ for some $\phi \in C_0(R^s)$ with $\hat{\phi}(0) \neq 0$. Then*

$$\mathcal{P} \subseteq \mathcal{S}_\phi \tag{2.4}$$

and the mapping

$$(Tf)(x) = \sum_{\alpha} f(\alpha)\phi(x - \alpha)$$

is one-to-one and onto $\mathcal{P}$.

Proof. Let $p \in \mathcal{P}$, and define $\psi(y) = p(y)\phi(x - y)$. For each x, $\psi \in C_0(R^s)$ and

$$\hat{\psi}(y) = e^{-ix\cdot y}\big(p(-iD + x)\hat{\phi}\big)(-y).$$

Hence Lemma 2.1 implies that

$$(Tp)(x) = \big(p(-iD + x)\hat{\phi}\big)(0). \tag{2.5}$$

Now, for each D, $q(x, D) = p(-iD + x) - p(x)$ is in $\mathcal{P}$ and $\deg q < \deg p$.

Thus $q(x; D)$ is a linear combination of elements in $\mathcal{P}$ with coefficients that are polynomials in D. But according to (2.5) we have

$$(Tp)(x) = \hat{\phi}(0)p(x) + \big(q(x; D)\hat{\phi}\big)(0) \tag{2.6}$$

From this equality it easily follows by induction on $\deg p$ that T is $1-1$ and onto $\mathcal{P}$. In fact, for each $n \geqslant 0$, T maps the class of polynomials with *exact* degree n in $\mathcal{P}$ onto itself. ∎

When $\mathcal{P} = \Pi_n$ this result appears in [8]; see also [9]. It is also known in this case that L^2 error estimates for approximating a smooth function by the scaled space

$$\mathcal{S}_{\phi,h} = \{S(h^{-1}x) : S \in \mathcal{S}_{\phi}\}, \qquad h > 0,$$

follow from Proposition 2.1. We will likewise obtain approximation results in the general context of Proposition 2.1. To this end, for any $\phi \in C_0(R^s)$, $\hat{\phi}(0) \neq 0$, we consider the power series expansion

$$[\hat{\phi}(x)]^{-1} = \sum_{\alpha} a_{\alpha} x^{\alpha}, \tag{2.7}$$

which is valid in some neighborhood of the origin. Let

$$(Lf)(x) = \sum_{\alpha} a_{\alpha}(-i)^{|\alpha|} D^{\alpha} f(x) \tag{2.8}$$

and

$$(Qf)(x) = \sum_{\alpha} (Lf)(\alpha)\phi(x-\alpha). \tag{2.9}$$

First let us observe that for *any* affinely invariant subspace $\mathcal{P}$ contained in $\mathcal{P}_{\phi}$ we have

$$Qp = p, \qquad p \in \mathcal{P}. \tag{2.10}$$

To prove this fact observe first that Lemma 2.2 implies $L(\mathcal{P}) \subseteq \mathcal{P}$ and so (2.5) gives

$$\begin{aligned}(Qp)(x) &= \sum_{\alpha} \sum_{\beta} \frac{(-i)^{|\alpha+\beta|}}{\beta!} a_{\alpha} (D^{\beta}\hat{\phi})(0) D^{\alpha+\beta} p(x) \\ &= \sum_{\gamma} (-i)^{|\gamma|} D^{\gamma} p(x) \left(\sum_{\alpha+\beta=\gamma} a_{\alpha} \frac{(D^{\beta}\hat{\phi})(0)}{\beta!} \right) \\ &= \sum_{\gamma} \delta_{0\gamma} (-i)^{|\gamma|} D^{\gamma} p(x) = p(x).\end{aligned}$$

To obtain approximation rates we let X be any normed linear function space on R^s with the property that translation is an isometry and restriction to closed subsets in R^s is a contraction. Extend the linear functional $(Lf)(0)$, $f \in \mathcal{P}$, to a linear functional F on X with support on $C = [-1,1]^s$, i.e.

$$\begin{aligned} |F(f)| &\leqslant \|F\| \, \|f_{|C}\|, \qquad f \in X, \\ F(f) &= (Lf)(0), \qquad f \in \mathcal{P}. \end{aligned} \tag{2.11}$$

For every $h > 0$, we set

$$(Q_h f)(x) = \sum_{\alpha} F(f(h(\cdot + \alpha)))\phi(h^{-1}x - \alpha).$$

Then it follows easily that for some constants c, r independent of h

$$|(Q_h f)(x)| \leqslant c \|f(h \cdot + x)_{|rC}\|. \tag{2.12}$$

Furthermore, using the affine invariance of $\mathcal{P}$, we also know for all h

$$Q_h p = p, \qquad p \in \mathcal{P}. \tag{2.13}$$

Hence we obtain

THEOREM 2.1. *Let $\phi \in C_0(R^s)$, and suppose $\mathcal{P}$ is an affinely invariant subspace of $\mathcal{P}_\phi$. Then there are constants c, r depending only on ϕ such that for all $h > 0$, $x \in R^s$,*

$$|f(x) - (Q_h f)(x)| \leqslant c \inf_{p \in \mathcal{P}} \|(f - p)(h \cdot + x)_{|rC}\|.$$

COROLLARY 2.1. *Let $\phi \in C_0(R^s)$, and suppose $\Pi_d \subseteq \mathcal{P}_\phi$. Then for some constant c, r independent of h and any $f \in W_p^d(R^s)$, one has for almost every $x \in R^s$*

$$|f(x) - (Q_h f)(x)| \leqslant c h^{d+1-s/p} \sum_{|\alpha| = d+1} \|D^\alpha f\|_{L^p(rhC + x)}. \tag{2.14}$$

REMARK 2.3. L^2 error estimates of the type given in (2.14) are given in [8].

3. SOME SPECIAL FUNCTIONS

In this section, we restrict ourselves to functions $\phi \in C_0(R^s)$ whose Fourier transform has the form

$$\hat{\phi}(x) = \hat{\rho}_1(x \cdot x^1) \cdots \hat{\rho}_n(x \cdot x^n), \tag{3.1}$$

each $\operatorname{supp} \rho_i$ compact. Our aim is to apply the result in Section 2 to this class of functions. For this purpose, we follow [3] and introduce a certain class of differential operators. First we set

$$\mathcal{K} = \{J: J \subseteq K, \langle K \setminus J \rangle \neq R^s\} \tag{3.2}$$

and then define

$$\mathcal{D} = \{f: f \in C^\infty(R^s), D_J f = 0 \text{ for all } J \in \mathcal{K}\}$$

where

$$D_J = \prod_{y \in J} D_y.$$

We also define

$$d + 1 = \min\{|J|: J \in \mathcal{K}\},$$

$|J| =$ cardinality of J (where we always count repeated vectors with their multiplicities).

Since we always require

$$\langle K \rangle = R^s, \tag{3.3}$$

in what follows, we see that $d \geqslant 0$. Moreover, since any $J \subseteq K$ with $|J| \geqslant n - s + 1$ is in $\mathcal{K}$, we also have $d \leqslant n - s$. Furthermore, it is obviously true that

$$\Pi_d \subseteq \mathcal{D}. \tag{3.4}$$

Somewhat less apparent is

PROPOSITION 3.1. *$\mathcal{D}$ is an affinely invariant subspace.*

Proof. The only claim that needs to be proved is the assertion

$$\mathcal{D} \subseteq \Pi.$$

To show this, let $f \in \mathcal{D}$; then $p(D)f = 0$ for any $p \in I$, the ideal generated by the polynomials

$$p_J(x) = \prod_{y \in J} x \cdot y \qquad J \in \mathcal{K}. \tag{3.5}$$

First we show that the only common zero of these polynomials in $\mathbb{C}^s$ is the zero vector. Thus we suppose $z \in \mathbb{C}^s$ and $p_J(z) = 0$, $J \in \mathcal{K}$. Pick any $J_0 \in \mathcal{K}$; then there is an $x^{i_1} \in J_0$ such that $z \cdot x^{i_1} = 0$. If $\langle x^{i_1} \rangle = R^s$, then $z = 0$ and we are finished. Otherwise, $K \setminus \{x^{i_1}\} \in \mathcal{K}$ and so there is an $x^{i_2} \notin \{x^{i_1}\}$ with $z \cdot x^{i_1} = z \cdot x^{i_2} = 0$. If $\langle x^{i_1}, x^{i_2} \rangle = R^s$, we get $z = 0$; otherwise, we repeat the process on the set $K \setminus \{x^{i_1}, x^{i_2}\}$. Eventually, the process terminates because we obtain $z \cdot x = 0$, $x \in J$, for some $J \subseteq K$ with $\langle J \rangle = R^s$. This finally implies $z = 0$. Now we can appeal to Hilbert's *Nullstellensatz* (see, e.g. [11, p. 5]) to conclude that for some positive integer $m > 0$ the ideal I contains the functions $x_1^m, \ldots, x_s^m$. Hence we conclude

$$\frac{\partial^m f}{\partial x_i^m} = 0, \qquad i = 1, \ldots, s,$$

which readily implies $f \in \Pi$. ∎

To apply Theorem 2.1 to the set $\mathcal{D}$ we need to find conditions on $\phi \in C_0(R^s)$ satisfying (3.1) which guarantee $\mathcal{D} \subseteq \mathcal{P}_\phi$. We will accomplish this by using the next lemma.

LEMMA 3.1. *Let $p \in \Pi$ and $f \in C^\infty(R^n)$. Then for any $x^1, \ldots, x^n \in R^s$ we have*

$$p(D)\big(f(x \cdot x^1, \ldots, x \cdot x^n)\big) = \sum_\alpha \frac{(D_{K^\alpha} p)(0)}{\alpha!} (D^\alpha f)(x \cdot x^1, \ldots, x \cdot x^n), \tag{3.6}$$

where

$$K^\alpha = \{\underbrace{x^1, \ldots, x^1}_{\alpha_1}, \ldots, \underbrace{x^n, \ldots, x^n}_{\alpha_n}\}, \qquad \alpha = (\alpha_1, \ldots, \alpha_n).$$

Proof. It suffices to prove this identity for $f(x) = e^{y \cdot x}$ where y is an arbitrary n-vector. In this case, (3.6) becomes

$$p\left(\sum_{i=1}^{n} y_i x^i \right) = \sum_{\alpha} \frac{(D_{K^\alpha} p)(0)}{\alpha!} y^\alpha. \tag{3.7}$$

Since $D^\alpha(p(\sum_{i=1}^n y_i x^i))|_{y=0} = (D_{K^\alpha} p)(0)$, we recognize (3.7) as the Taylor expansion of $p(\sum_{i=1}^n y_i x^i)$ at the origin, which establishes the result. ∎

This lemma gives us

THEOREM 3.1. *Let $\phi \in C_0(R^s)$ such that*

$$\hat{\phi}(x) = \prod_{i=1}^{n} \hat{\rho}_i(x \cdot x^i),$$

where each $\operatorname{supp} \rho_i$ *is compact and* $\hat{\rho}_i(2\pi j) = 0$, $j \in Z - \{0\}$. *Then for any set* $K = \{x^1, \dots, x^n\} \subseteq Z^s$ *we have*

$$\mathfrak{D} \subseteq \mathcal{P}_\phi.$$

Proof. Lemma 3.1 implies that for any $p \in \Pi$

$$(p(D)\hat{\phi})(x) = \sum_{\beta} \frac{(D_{K^\beta} p)(0)}{\beta!} \omega_\beta(x),$$

where we set

$$\omega_\beta(x) = \prod_{i=1}^{n} \hat{\rho}_i^{(\beta_i)}(x \cdot x^i), \qquad \beta = (\beta_1, \dots, \beta_n).$$

Now, suppose $x = 2\pi\alpha$, $\alpha \in Z^s - \{0\}$. For any β having the property

$$J_\beta = \{x^i : \beta_i \neq 0\} \notin \mathcal{K}$$

we know that at least one of the numbers $\alpha \cdot y$, $y \notin J_\beta$, is nonzero. Hence $\omega_\beta(2\pi\alpha) = 0$ in this case. Thus

$$(p(D)\hat{\phi})(2\pi\alpha) = \sum_{J_\beta \in \mathcal{K}} \frac{(D_{K^\beta} p)(0)}{\beta!} \omega_\beta(x). \tag{3.8}$$

However, for any β, $J_\beta \subseteq K^\beta$, and so when $p \in \mathfrak{D}$ all summands in (3.8) are zero and the claim is proven. ■

As we will refer to $\phi \in C_0(R^s)$ which satisfy the hypothesis of Theorem 3.1 frequently, we will denote this class by Λ.

From Propositions 2.1 and 3.1 and Theorem 3.1 we conclude

COROLLARY 3.1. *For any $\phi \in \Lambda$, we have*

$$\mathfrak{D} \subseteq \mathcal{S}_\phi.$$

Choosing $\phi(x) = B(x|x^1,\dots,x^n)$ allows us to sharpen Proposition 3.1:

$$\Pi_d \subseteq \mathfrak{D} \subseteq \Pi_{n-s}.$$

COROLLARY 3.2. *For any $\phi \in \Lambda$ and $f \in W_p^k(R^s)$, $k \geqslant d+1$,*

$$\inf_{S \in \mathcal{S}_{\phi,h}} \|f - S\|_{L^\infty(R^s)} = O(h^k).$$

Actually, Corollary 3.2 can be improved. We will give a *saturation theorem* for a subset of functions in Λ. This requires improved local approximation estimates for the class $\mathfrak{D}$. For this purpose, we introduce the following additional notation.

We define $\mathcal{K}_c$ as the set of all elements in $\mathcal{K}$ not containing, as a proper subset, a set in $\mathcal{K}$. Hence it follows that

$$\mathfrak{D} = \{f \colon f \in C^\infty(R^s),\, D_J f = 0,\, J \in \mathcal{K}_c\}.$$

Moreover, let $m = \max\{|J| \colon J \in \mathcal{K}_c\}$ so that $d+1 \leqslant m \leqslant n-s+1$. In addition we introduce classes

$$\mathfrak{D}_l = \left\{f \colon f \in W_p^l(R^s),\, D_J f = 0,\, J \in \mathcal{K}_c,\, |J| = l\right\}$$

for $d+1 \leqslant l \leqslant m$, $1 < p < \infty$.

LEMMA 3.2. *There exists a constant c such that for every $f \in W_p^m([-1,1]^s)$, $1 < p < \infty$, we have*

$$\min_{q \in \mathfrak{D}} \|f - q\|_{L^p([-1,1]^s)} \leqslant c \sum_{l=d+1}^{m} \left(\sum_{|J| = l,\, J \in \mathcal{K}_c} \|D_J f\|_{L^p([-1,1]^s)} \right).$$

Proof. The proof of Proposition 3.1 shows that the polynomials

$$p_J(x) = \prod_{y \in J} x \cdot y, \qquad J \in \mathcal{K}_c$$

have $z = 0$ as the only common zero in $\mathbf{C}^s$. Thus the coercive estimate of K. T. Smith [10] implies that for any constant $a > 0$ the set of all f such that

$$\|f\|_{L^p([-1,1]^s)} \leqslant a$$

and

$$\|D_J f\|_{L^p([-1,1]^s)} \leqslant a, \qquad J \in \mathcal{K}_c,$$

is precompact in $L^p([-1,1]^s)$. Therefore, by a standard argument, the lemma follows. ■

We can now combine Theorem 3.1 and Lemma 3.2 to get

THEOREM 3.2. *Let $\phi \in \Lambda$, and suppose each ρ_i in (3.1) is a step function. Then for $f \in W_p^m(R^s)$, $1 < p < \infty$, $d + 1 \leqslant l \leqslant m$, we have*

$$\inf_{S \in \mathcal{S}_{\phi,h}} \|f - S\|_{L^p(R^s)} = o(h^l) \tag{3.9}$$

if and only if

$$f \in \bigcup \{\mathcal{D}_k : d + 1 \leqslant k \leqslant l\}. \tag{3.10}$$

Proof. The sufficiency of the condition (3.10) follows from Corollary 3.2. The first step in the proof of the necessity is to observe that "locally" every $S \in \mathcal{S}_{\phi,h}$ is in $\mathcal{D}$. This follows because our hypothesis implies ϕ is a sum of box splines. Since the box splines are "locally" in $\mathcal{D}$, so too is any $S \in \mathcal{S}_{\phi,h}$. Alternatively, we can return to (3.1) and use integration by parts to verify that the distribution

$$\int_{R^s} \phi(x) D_J f(x)\, dx$$

for $J \in \mathcal{K}$ is a sum of distributions supported on hyperplanes parallel to the subspace spanned by $K \setminus J$. As a consequence, any $S \in \mathcal{S}_{\phi,h}$, restricted to a region G not intersected by these hyperplanes for any J, is in $\mathfrak{D}$. In particular, we have

$$\mathcal{S}_{\phi,h} \cap \Pi = \mathfrak{D}.$$

To finish the proof, we fix any point $x \in G$. We set

$$\Delta_y f(x) = f(x+y) - f(x)$$

and define $\Delta_{hJ} = \prod_{y \in J} \Delta_{hy}$. There is a $p \in \mathfrak{D}$ such that $|f(y) - p(y)| = o(h^l)$ for $y \in G$. Since

$$(\Delta_{hJ} p)(x) = \int_{R^s} B(x - y|hJ)(D_J p)(y)\, dy = 0,$$

we get

$$(\Delta_{hJ} f)(x) = o(h^l)$$

for h sufficiently small. Hence for $|J| \leqslant l$ we conclude $D_J f = 0$ a.e., which finishes the proof. ■

We end this section with the following result.

PROPOSITION 3.2. *Suppose $\phi \in \Lambda$ and each ρ_i has a simple zero at each $2\pi j$, $j \in Z - \{0\}$. Then*

$$\mathcal{P}_\phi \cap \Pi_{d+1} \subseteq \mathfrak{D} \cap \Pi_{d+1}.$$

Proof. Let $p \in \mathcal{P}_\phi \cap \Pi_{d+1}$. Then, according to (3.4), we can assume $\deg p = d+1$. Choose any $J \in \mathcal{K}$. We will show that $D_J p = 0$. Thus we can also assume $|J| = d+1$. In this case, $D_J p(x)$ is a constant, and so it suffices to show $D_J p(0) = 0$. To this end, we choose an $\alpha^o \in Z^s - \{0\}$ such that $\alpha^o \cdot x = 0$, $x \notin J$. Thus from (3.8) we have

$$\big(p(D)\hat{\phi}\big)(2\pi\alpha^o) = \sum_{\substack{J \in \mathcal{K} \\ |\beta|^\beta \leqslant d+1}} \frac{(D_{K^\beta} p)(0)}{\beta!} \omega_\beta(2\pi\alpha^o).$$

Since any $J \subseteq K$, $|J| \leqslant d$ is not in $\mathcal{K}$, we can sum only over $|\beta| = d+1$.

Furthermore, if there is an i such that $x^i \notin J$ and $\beta_i \neq 0$, then $\omega_\beta(2\pi\alpha^o) = 0$. Thus there is only one summand above corresponding to $\beta_i = 1$, $x^i \in J$, and $\beta_i = 0$, $x^i \notin J$. Consequently, we get

$$(p(D)\hat{\phi})(2\pi\alpha^o) = (D_J p)(0) \prod_{x^i \notin J} \hat{\rho}_i(0) \prod_{x^i \in J} \hat{\rho}_i'(2\pi\alpha^o \cdot x).$$

Finally, since $p \in \mathcal{P}_\phi$, we conclude that $D_J p(0) = 0$, which proves the result. ∎

4. LINEAR INDEPENDENCE OF TRANSLATES

In this section, we are primarily concerned with showing the linear independence of the translates of the box spline. First though, we will make some general observations about linear independence of translates of a function $\phi \in C_0(R^s)$.

It is easy to see, by using the Fourier transform of ϕ, that

$$\sum_\alpha a_\alpha \phi(x - \alpha) = 0, \qquad x \in R^s, \tag{4.1}$$

implies $a_\alpha = 0$, $\alpha \in Z^s$, when $\sum_\alpha |a_\alpha| < \infty$. On the other hand, it was shown in [2] for $a_\alpha = p(\alpha)$, $p \in \Pi$, that (4.1) also implies $a_\alpha = 0$, $\alpha \in Z^s$. Each of these observations show that some "growth condition" on any sequence $\{a_\alpha\}$ satisfying (4.1) forces $a_\alpha = 0$, $\alpha \in Z^s$. The next result says that linear independence of the translates for any $\phi \in C_0(R^s)$ is governed by "exponentially growing" sequences.

THEOREM 4.1. *Let $\phi \in C_0(R^s)$. Then the translates of ϕ are linearly independent if and only if there does not exist any $z \in (\mathbf{C} - \{0\})^s$ such that*

$$\sum_\alpha z^\alpha \phi(x - \alpha) = 0, \qquad x \in R^s.$$

Proof. We assume, without loss of generality, that $\operatorname{supp} \phi \subseteq [0, N]^s$. Then for $x \in \operatorname{supp} \phi$,

$$p_x(z) = z^N \sum_\alpha z^\alpha \phi(x + \alpha) \tag{4.2}$$

is a polynomial in z. If $\sum_\alpha a_\alpha \phi(x+\alpha)=0$ for all $x \in R^s$, then

$$p_x(E)a=0,$$

where $E=(E_1,\dots,E_s)$ and $E_i f(x)=f(x+e^i)$, $(e^i)_j=\delta_{ij}$. Thus $p(E)a=0$ for all p in the ideal I generated by the polynomials (4.2). By hypothesis any common zero $z=(z_1,\dots,z_s)$ of these polynomials satisfies $z_1\cdots z_s=0$. Hence by Hilbert's *Nullstellensatz* we conclude that for some $k\geqslant 0$, $(z_1\cdots z_s)^k \in I$. Therefore $E^k a=0$ and so $a=0$. ∎

More specifically, for $\phi \in \Lambda$, we follow [3], and note the following necessary condition for linear independence of the translates. Given any $J \subseteq K$, $|J|=s$, $\langle K\rangle = R^s$, we consider the transformation

$$x=A_J u=\sum_{i=1}^{s} u_i x^{j_i}, \qquad J=\{x^{j_1},\dots,x^{j_s}\}.$$

Then for $\psi(u)=\phi(x)$ we obtain

$$\hat{\psi}(u)=|\det A_J|^{-1}\hat{\phi}\Big(\big(A_J^{-1}\big)^T u\Big)$$

and so

$$\sum_\beta \phi(x-A_J\beta)=\sum_\beta \psi(u-\beta)=\hat{\psi}(0)=\frac{\prod_{i=1}^{n}\hat{\rho}_i(0)}{|\det A_J|}.$$

Thus a necessary condition for linear independence of $\phi(\cdot-\alpha)$, $\phi\in\Lambda$, is that

$$|\det A_J|=1 \tag{4.3}$$

whenever J is a basis for R^s in K. We now show that (4.3) is also sufficient for the linear independence of the translates of the box spline.

THEOREM 4.2. *Given any* $K=\{x^1,\dots,x^n\}\subseteq Z^s$, $\langle K\rangle=R^s$, *such that* (4.3) *holds. Whenever*

$$\sum_\alpha a_\alpha B(x-\alpha|x^1,\dots,x^n)=0, \qquad x\in R^s,$$

it follows that $a_\alpha=0$, $\alpha\in Z^s$.

Proof. For any $f \in C_0^\infty(R^s)$ and $J \in \mathcal{K}$, $|J| = l$,

$$0 = \int_{R^s} \sum_\alpha a_\alpha B(x - \alpha | x^1, \ldots, x^n) D_J f(x)\, dx$$

$$= \int_{[0,1]^{n-l}} \sum_\alpha (p_J(E^{-1})a)(\alpha) f\Big(\sum_{x^i \notin J} \lambda_i x^i + \alpha \Big)\, d\lambda,$$

where $p_J(E^{-1})$ is the difference operator

$$p_J(E^{-1}) = \prod_{y \in J} (1 - E^{-y}). \tag{4.4}$$

Hence we easily conclude that

$$p_J(E)a = 0, \qquad J \in \mathcal{K}. \tag{4.5}$$

Replacing the box spline by one of its translates, we may assume $K \subseteq Z_+^s$. Thus p_J is a polynomial and so

$$p(E)a = 0 \tag{4.6}$$

for any p in the ideal generated by p_J, $J \in \mathcal{K}$. Just as in the proof of Proposition 3.1, we can show that the only common zero of these polynomials satisfies

$$z^x = 1, \qquad x \in J, \tag{4.7}$$

for some $J \subseteq K$ such that $\langle J \rangle = R^s$. Using (4.3), we conclude easily that $z = (1, 1, \ldots, 1)$. Therefore the *Nullstellensatz* implies that for some positive integer $m > 0$, the polynomials $(z_1 - 1)^m, \ldots, (z_s - 1)^m$ are in the ideal I generated by p_J, $J \in \mathcal{K}$. Therefore $a_\alpha = q(\alpha)$ for some $q \in \Pi$. Again we follow [3] to show $q \in \mathcal{D}$. The argument is

$$0 = p_J(E)q(x) = \int_{[0,1]^{|J|}} (D_J q)\Big(\sum_{x^i \in J} \lambda_i x^i + x \Big)\, d\lambda$$

for all x. Therefore $D_J q = 0$ for all $J \in \mathcal{K}$. Since we now know $q \in \mathcal{D}$, we can invoke Proposition 2.1 and conclude $q = 0$. This finishes the proof of the theorem. ∎

COROLLARY 4.1. *Let $1 \leqslant p \leqslant \infty$. Then there exist constants $\underline{c}, \bar{c} > 0$ such that*

$$\underline{c}\|a\|_{l^p} \leqslant \left\| \sum_\alpha a_\alpha h^{-s/p} B(h^{-1}\cdot - \alpha | K) \right\|_{L^p(R^s)} \leqslant \bar{c}\|a\|_{l^p}.$$

The proof of this result follows easily from Theorem 4.2 and the fact that $P(x|x^1,\ldots,x^n)$ has compact support.

FINAL REMARKS.

(i) The assumption $\phi \in C_0(R^s)$ was chosen to simplify the presentation. It is easily weakened to continuity almost everywhere. In this case, if $\alpha \in Z^s$ happens to be a point of discontinuity of ϕ, we simply define $\phi(\alpha)$ to be the limit obtained by the mollifying process in Lemma 2.1.

(ii) Likewise, Proposition 3.1 remains valid when derivatives are interpreted in a weak sense and $f \in L^1_{\text{loc}}(R^s)$.

Note added in proof: Rong-qing Jia also proved Theorem 4.2 independently. His paper, "Linear Independence of Translates of a Box Spline," will appear in the *Journal of Approximation Theory*. For further comments and consequences on the method used here see, W. Dahmen and C. A. Micchelli, Recent Results on Multivariate Splines, *Proceedings of the International Conference in Approximation Theory, Texas A & M University, January 1983*, Academic Press, New York.

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